AIAA JOURNAL
Vol. 43, No. 6, June 2005

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Time Decay of *n* Family of Vortices

Georgios H. Vatistas,* Yasser Aboelkassem,† and M. H. Kamran Siddiqui‡
Concordia University, Montreal, Quebec H3G 1M8, Canada

I. Introduction

OVER a decade ago, Vatistas et al.[1] proposed a family of disingularized algebraic vortex formulations. The most often used member of the n family is $n = 2$. The reasons for this preference lie in the fact that $n = 2$ is mathematically simple (thus, computationally efficient), the predicted tangential velocity distribution is in good agreement with the experimental data, and the pressure is given explicitly. Because of these characteristics, the formulation has been employed in several studies. (See, for example, Refs. 2–5.)

This Note focuses on the extension of the n family of vortices[1] into time decay. In the course of broadening the model, we have formalized a space–time analogy and have elaborated on some of its unique properties. Similar to the steady companion, the present analysis is limited to high Reynolds number vortices.

II. Formulation of Problem

Consider the motion of an incompressible, tubular, intense vortex. The equations employed to describe this flow are the axisymmetric Navier–Stokes in cylindrical coordinates for continuity

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} = 0$$

radial momentum

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial \Delta P}{\partial r} + \nu\left\{\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{\partial^2 V_r}{\partial z^2}\right\}$$

tangential momentum

$$\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r} = \upsilon\left\{\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} + \frac{\partial^2 V_\theta}{\partial z^2}\right\}$$

and axial momentum

$$\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + V_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial \Delta P}{\partial z} + \nu\left\{\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial z^2}\right\}$$

where r, θ and z, are the radial, azimuthal, and axial coordinates; t is the time; V_r, V_θ, and V_z are the radial, tangential, and axial velocity components; $\Delta P = p - p_\infty$, p is the static pressure; p_∞ is the value of the pressure far from the vortex center ($r \to \infty$); ρ is the density; and ν is the kinematic viscosity.

We are interested in solutions that belong to a particular class in which the velocity vector has the general form

$$\boldsymbol{V}[u(\tau,\xi),\, v(\tau,\xi),\, \zeta h(\tau,\xi)]$$

where $\tau = \nu t/r_c^2$, $\xi = r/r_c$, $\zeta = z/r_c$, $u(\tau,\xi) = v_r/v_{\theta c}$, $w(\tau,\xi) = v_z/v_{\theta c} = \zeta h(\tau,\xi)$, $v(\tau,\xi) = v_\theta/v_{\theta c}$, r_c is the core size, $v_{\theta c} = \kappa/r_c$, $\kappa = \Gamma_\infty/2\pi$, and Γ_∞ is the vortex circulation. The subscript c denotes value of the parameter at the core (defined as the radius where the tangential velocity attains its maximum).

The equations governing this particular flowfield, in dimensionless form, are continuity

$$\underset{\delta}{\frac{\partial u}{\partial \xi}} + \underset{\delta}{\frac{u}{\xi}} + \underset{\delta}{h} = 0 \tag{1}$$

radial momentum

$$\underset{\delta}{\frac{\partial u}{\partial \tau}} + \underset{1/\delta}{Re}\left\{\underset{(\delta\ \delta}{u\frac{\partial u}{\partial \xi}} - \underset{1)}{\frac{v^2}{\xi}}\right\} = -\underset{1/\delta}{Re}\underset{\delta}{\frac{\partial \Delta p}{\partial \xi}} + \underset{\delta}{\frac{\partial^2 u}{\partial \xi^2}} + \underset{\delta}{\frac{1}{\xi}\frac{\partial u}{\partial \xi}} - \frac{u}{\xi^2} \tag{2}$$

tangential momentum

$$\underset{1}{\frac{\partial v}{\partial \tau}} + \underset{1/\delta}{Re}\left\{\underset{\delta 1}{u\frac{\partial v}{\partial \xi}} + \underset{1\delta}{\frac{vu}{\xi}}\right\} = \underset{1}{\frac{\partial^2 v}{\partial \xi^2}} + \underset{1}{\frac{1}{\xi}\frac{\partial v}{\partial \xi}} - \underset{1}{\frac{v}{\xi^2}} \tag{3}$$

and axial momentum

$$\underset{\delta}{\frac{\partial h}{\partial \tau}} + \underset{1/\delta}{Re}\left\{\underset{(\delta\ \delta}{u\frac{\partial h}{\partial \xi}} + \underset{\delta^2)}{h^2}\right\} = -\underset{1/\delta}{\frac{Re}{\xi}}\frac{\partial \Delta p}{\partial \xi} + \underset{\delta}{\frac{\partial^2 h}{\partial \xi^2}} + \underset{\delta}{\frac{1}{\xi}\frac{\partial h}{\partial \xi}} \tag{4}$$

where $\Delta p(\tau,\xi) = \Delta P/\rho v_{\theta c}^2$, $Re = v_{\theta c} r_c/\nu$ is the vortex Reynolds number, and δ is a very small number.

Because we are dealing with strong vortices, we adopt the traditional assumption that requires that u and $h \ll v$. In terms of order of magnitude, if v is of $\mathcal{O}$ (1), then u and h are of $\mathcal{O}$ (δ), where $\delta \sim 1/Re$. (Note that for aerodynamic vortices $Re \sim 10^5$.)

Using order of magnitude considerations, one can bring the preceding system of equations into a simpler form. Because all of the terms in Eqs. (1) and (3) are of the same order, both equations remain unchanged. Neglecting the terms of order δ in Eq. (2), we obtain radial momentum

$$\frac{v^2}{\xi} = \frac{\partial \Delta p}{\partial \xi} \tag{5}$$

The axial momentum indicates that the static pressure should not vary appreciably in the ζ direction:

$$Re\frac{\partial \Delta p}{\partial \zeta} \sim \delta, \qquad \text{or} \qquad \frac{\partial \Delta p}{\delta \zeta} \sim \delta^2 \to \frac{\partial \Delta p}{\delta \zeta} \approx 0 \tag{6}$$

The set that consists of Eqs. (1), (3), (5), and (6) is the one that produced the classical vortices of Lamb (see Ref. 6), Burgers,[7] and Sullivan.[8]

Received 4 September 2004; revision received 21 December 2004; accepted for publication 21 December 2004.

*Professor, Department of Mechanical and Industrial Engineering. Senior Member AIAA.

†Research Assistant, Department of Mechanical and Industrial Engineering.

‡Assistant Professor, Department of Mechanical and Industrial Engineering.

The required initial and boundary conditions are as follows:

1) First, $\tau = \tau_{in}$, $u(\tau_{in}, \xi) = f_1(\xi)$, $h(\tau_{in}, \xi) = f_2(\xi)$, and $v(\tau_{in}, \xi) = f_3(\xi)$.

2) Second, $\xi = 0$, $v(\tau, \xi) = u(\tau, \xi) = 0$, and $\partial h(\tau, \xi)/\partial \xi = 0$.

3) Third, $\xi \to \infty$ and $v(\xi)\xi \to 1$.

Under the following variable transformation relations:

$$\eta = \xi\big/\sqrt{\tau}, \qquad V(\eta) = v(\tau, \xi)\sqrt{\tau}$$

$$U(\eta) = u(\tau, \xi) Re\sqrt{\tau} - \eta/2$$

$$H(\eta) = 1 + h(\tau, \xi) Re\tau, \qquad \Delta\Pi(\eta) = \Delta p(\tau, \xi)\tau \tag{7}$$

the partial differential equations (1), (3), and (5) convert into the following ordinary set:

$$\eta^{-1}|U\eta|' + H(\eta) = 0 \tag{8}$$

$$\eta^{-1}V^2 = \Delta\Pi' \tag{9}$$

$$V'' + \eta^{-1}(1 - U\eta)V' - \eta^{-2}(1 + U\eta)V = 0 \tag{10}$$

subject to the boundary conditions:

1) First, $\eta = 0$, $V = U = 0$, and $H' = 0$.

2) Second, $\eta \to \infty$ and $V\eta \to 1$.

The primes represent differentiation with respect to η.

The steady-state subset is then

$$\xi^{-1}(u\xi)' + h = 0 \tag{11}$$

$$\xi^{-1}v^2 = \Delta p' \tag{12}$$

$$v'' + \xi^{-1}(1 - u\xi)v' - \xi^{-2}(1 + u\xi)v = 0 \tag{13}$$

which respects the following boundary conditions:

1) First, $\xi = 0$, $v = u = 0$, and $h' = 0$.

2) Second, $\xi \to \infty$ and $v\,\xi \to 1$.

In this case, the primes represent differentiation with respect to ξ.

It is clear that the two systems given by Eqs. (8–10) and (11–13) are dual, which further suggests that if a steady vortex is known, the corresponding decaying version can be obtained through a straightforward variable transformation and vice versa.

III. Results and Discussion

The modus operandi of the transformation will be shown next through the conversion of the steady $n = 2$ into time decaying. The velocity components and the static pressure for this stationary vortex are[1,9]

$$u(\xi) = -\frac{6\xi^3}{Re(1+\xi^4)}, \qquad v(\xi) = \frac{\xi}{\sqrt{1+\xi^4}}$$

$$h(\xi) = \frac{24\xi^2}{Re(1+\xi^4)^2}, \qquad \Delta p(\xi) = \frac{1}{2}\left\{\operatorname{arc\,tg}(\xi^2) - \frac{\pi}{2}\right\}$$

Because of the analogy of the two systems (steady and unsteady), the transformed velocity and pressure expressions are, thus,

$$U(\eta) = -\frac{6\eta^3}{Re(1+\eta^4)}, \qquad V(\eta) = \frac{\eta}{\sqrt{1+\eta^4}}$$

$$H(\eta) = \frac{24\eta^2}{Re(1+\eta^4)^2}, \qquad \Delta\Pi(\eta) = \frac{1}{2}\left\{\operatorname{arc\,tg}(\eta^2) - \frac{\pi}{2}\right\} \tag{14}$$

It is easy to verify that the preceding expressions do indeed satisfy Eqs. (8–10).

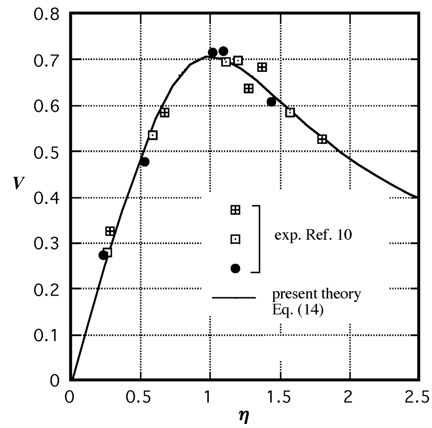

Fig. 1 Experimental verification of analogy.

The preceding fluid properties can now be given as functions of the time and the radius:

$$u(\tau, \xi)Re = \frac{1}{2}\left\{\frac{\xi}{\tau} - \frac{12\xi^3}{\tau^2 + \xi^4}\right\}, \qquad v(\tau, \xi) = \frac{\xi}{(\tau^2 + \xi^4)^{\frac{1}{2}}}$$

$$h(\tau, \xi)Re = \frac{24\xi^2\tau^2}{(\tau^2 + \xi^4)^2} - \frac{1}{\tau}$$

$$\Delta p(\tau, \xi) = \frac{1}{2\tau}\left\{\operatorname{arc\,tg}\left(\frac{\xi^2}{\tau}\right) - \frac{\pi}{2}\right\} \tag{15}$$

The formulas of Eq. (15) obey Eqs. (1), (3), (5), and (6) and also respect the associated boundary conditions. When the profiles given by Eq. (15) at $\tau = \tau_{in} = 1$ are taken to represent the dependent variables at the start of the decaying process, they will also automatically satisfy the initial conditions. It is amply evident that as $\tau \to \infty$ all of the velocity components tend to zero, whereas the pressure approaches the constant ambient value p_∞.

The variable transformation relations given in Eq. (7) suggest that the velocity and the pressure should be functions of η only. The latter points out further that, under the variable transformation, the individual time-dependent variables given by Eq. (15) must collapse into single curves for U, V, and H. The last is validated for the tangential velocity in Fig. 1 by using the experimental results of Bennet.[10] Unfortunately, no experimental data for the static pressure are available, making a similar comparison impossible.

The other members of the n family can be transformed in the same manner into time dependence. Similarly, the topology can be used to convert steady-state models such as those of Sullivan[8] and Vatistas[9] into their decaying counterparts. Reversing the process, one can recover the time-invariant group that gives rise to the diffusing Bellamy-Knights[11] set of vortices. Because the last work is based on Sullivan's contribution, it will also provide steady solutions of Sullivan-like vortices other than the original Sullivan steady vortex. Furthermore, it is not difficult to prove, by the use of the present methodology, that the Lamb vortex will give rise to Burgers steady vortex, and vice versa. The latter is already known, but it is mentioned here to fortify further the validity of the method.

IV. Conclusions

In this Note, we present a methodology to extend the n family of algebraic vortices[1] to include the temporal effects. The developed method can also be used to convert other steady vortex formulations (of the same kind) into unsteady vortices and vice versa. The theoretical development presupposed intense conditions, and, as such, the method is only applicable to high Reynolds number vortices.

References

[1] Vatistas, G. H., Kozel, V., and Minh, W., "A Simpler Model for Concentrated Vortices," *Experiments in Fluids*, Vol. 11, No. 1, 1991, pp. 73–76.

[2]Nilay, S. U., "High-Accuracy Wake and Vortex Simulations Using a Hybrid Euler/Discrete Vortex Method," M.S. Thesis, Dept. of Aerospace Engineering, Pennsylvania State Univ., University Park, PA, May 2001.

[3]Jeng-Lih, H., Jing-Fa, T., Chye-Horng, C., and Cheng-Tsair, Y., "Measurements of Propeller Tip Vortices Around Near Wake Zone," *Ship Technology Research*, Vol. 46, No. 2, 1999, pp. 93–110.

[4]Leishman, J. G., "Measurements of the Aperiodic Wake of a Hovering Rotor," *Experiments in Fluids*, Vol. 125, No. 4, 1998, pp. 252–361.

[5]Tauszig, L., "Numerical Detection and Characterization of the Blade Vortex Interaction Using Free Wake Analysis," M.S. Thesis, Dept. of Aerospace Engineering, Pennsylvania State Univ., University Park, PA, Aug. 1998.

[6]Lamb, H., *Hydrodynamics*, 6th ed., Cambridge Univ. Press, Cambridge, England, U.K., 1932, pp. 592, 593.

[7]Burgers, J. M., "A Mathematical Model Illustrating the Theory of Turbulence," *Advances in Applied Mechanics*, Vol. 1, 1948, pp. 171–199.

[8]Sullivan, R. D., "A Two-Cell Vortex Solution of the Navier–Stokes Equations," *Journal of the Aerospace Sciences*, Vol. 26, No. 11, 1959, pp. 767, 768.

[9]Vatistas, G. H., "New Model for Intense Self-Similar Vortices," *Journal of Propulsion and Power*, Vol. 14, No. 14, 1998, pp. 462–469.

[10]Bennet, T. J., "Vortex Coalescence and Decay," Ph.D. Dissertation, Dept. of Civil and Environmental Engineering, Washington State Univ., Pullman, WA, Dec. 1988.

[11]Bellamy-Knights, P. G., "An Unsteady Two-Cell Vortex Solution of the Navier–Stokes Equations," *Journal of Fluid Mechanics*, Vol. 41, No. 3, 1970, pp. 673–687.

A. Plotkin
Associate Editor